Elements of the Calculus of Variations *

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Soon after the invention of the principles of differential calculus mathematicians have started to consider problems requiring a completely special application of this calculus. For, since the main essence of differential calculus is that, having propounded an arbitrary function of the variable quantity x, its increment is investigated, if the quantity x is assumed to increase by the amount of its differential dx, after a translation to Geometry it was easy to define the tangents and curvatures of curved lines, which quantities are immediately derived from the nature of differentials. Another matter is the nature of problems, in which innumerable curved lines contained in a general equation are propounded, from which curves equally long arcs, or such arcs which are traveled along in the same amount of time by a body only subjected to gravity, are to be separated, from the latter of which cases the problem of synchronous curves originated. For, in questions of this kind it is not so important how much the ordinate of a certain curve increases if the abscissa is increased by its differential, but rather how the arc length or the time of descent is varied by the differential, if the arc is taken on another curve. Such problems are said to be resolved by differentiation of parameters, since the variability of the parameter contains all the infinitely many propounded curves. But to see the principles, from which the solution of problems of this kind is to be derived, more clearly, let an arbitrary equation among the abscissa x and the ordinate y be propounded, which we additionally want to contain the constant a to be referred to as parameter; as long as it retains the same value, the equation will

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yield one single curved line, but if successively different values are attributed to a, other curved lines will arise. If the question is about the arcs of these curves, since the arc of a curve is expressed by $\int \sqrt{dx^2 + dy^2}$, in which integration the parameter a is treated as a constant, the whole task reduces to the definition of the increment of the integral formula $\int \sqrt{dx^2 + dy^2}$ it receives, if in it the quantity $a + da$ is substituted for a. Therefore, in general, if any other integral expression $\int Zdx$ instead of the arc is considered, which integration is to be constructed from the equation given among x and y, having treated the parameter a as a constant, it is in question, how much of a variation the same expression $\int Zdx$ will undergo, if in the equation given among x and y the parameter a is increased by its differential da. The famous problem of the orthogonal trajectories is of the same nature, in which infinitely many curved lines given by an equation among the abscissa x, the ordinate y and the parameter a are propounded and a curved line is in question, which intersects all those curves orthogonally. To solve this problem the ordinate y is usually considered as a function of x and a, from whose differentiation the form $dy = pdx + qda$ is assumed to result; but then one gets to this differential equation

$$dx(1 + pp) + pqda = 0 \quad \text{or to this one} \quad dx + pdy = 0,$$

from which in combination with the first the parameter a must be eliminated in order to find an equation among x and y expressing the nature of the curve in question. Whenever the curve to be intersected is given through an algebraic equation among x and y, the task is simple, since hence the value of y can be defined in terms of x and a absolutely and hence the values of p and q can be assigned by differentiation, whence a differential equation among the two variables p and q only is obtained; but if the equation for the curves to be intersected is a differential equation involving the parameter a as a constant quantity, which equation will therefore be of the form $dy + pdx$ or $y = \int pdx$, one especially has to investigate, a differential equation of which kind would have resulted, if, except for x, also the parameter a is considered as a variable, in order to find the quantity q from this; this investigation becomes very difficult in most cases and hence seems to exceed the possibilities of analysis. But even though the reasoning for this investigation is to be derived solely from the principles of differential calculus, there is nevertheless a huge difference in the application, since, whereas ordinary differentiation usually is not difficult at all, here the whole difficulty resides in the invention of the differentials to arise from the variability of the parameter and this invention requires own special rules. Therefore, the branches of the Analysis of the Infinite necessarily seem to be increased, if we refer an investigation of differentials of this kind, which result from the variability of the parameter, to a peculiar calculus, which for the sake of distinction can be called *Calculus of Variations*. The necessity for it will be seen even more clearly, if we consider that it extends a lot further than just to the variability of the parameter; even though by that parameter the amount of curves is multiplied to infinity, all of them are nevertheless comprehended in a certain species, which is contained in the given equation, of course. But our calculus of variations can not only be extended to certain classes of curves of this kind, but even to all curves one can think of, as, e.g., if among all curves the one is to be defined which enjoys a certain given property of maximum or minimum. And the famous isoperimetric problem understood in the broadest sense, as I considered it in my book on the calculus of variations, is to be referred to this; who read my book with attention will not doubt that investigations of this kind require a singular calculus not very different from the usual rules of Analysis. For, these problems are reduced to such a question, that an equation among the two variables x and y is determined, from which a certain integral expression $\int Zdx$, no matter how Z depends on x and y, obtains a maximum or minimum value. To achieve this, having propounded an arbitrary integral formula $\int Zdx$ which obtains a determined value from the assumed relation among x and y, it is necessary to define in general, how much of an alteration the formula will undergo, if the relation among x and y is varied infinitely less; and this question extends infinite times further than the above question, where only the change to arise from the variation of the parameter had to be assigned. But instead of this simple integral formula $\int Zdx$ one can also consider any expression composed of x, y and their differentials and integral formulas, to extend this subject even further; even then the calculus of variations will yield rules to define the change of expressions of this kind induced by an infinitely small variation of the given relation among x and y. The method usually applied for the solution of isoperimetric problems indeed already provides us with extraordinary specimens of this calculus; but since they are all taken from the same source, i.e. Geometry, they can not be used for the foundation of the principles of this desired calculus. Furthermore, even these specimens are not general enough to comprehend the girth of our calculus. Therefore, I decided to derive its elements from the first principles of Analysis and expand them in such a way that they can not only be applied to solve the above problems quickly and easily but also open a new field extending to many other questions of this kind, in which Mathematicians can then test their abilities while promoting the limits of Analysis tremendously.

HYPOTHESIS 1

§1 Let an arbitrary equation among the two variables x and y be given, which also expresses their mutual relation, such that hence, whatever determined value is attributed to x, also a determined value for y is defined.

COROLLARY 1

§2 Therefore, having propounded an equation among the two variables x and y to all conceivable values of x determined values of y will correspond.

COROLLARY 2

§3 Therefore, via this propounded equation y will be a certain function of x and, as y corresponds to x, so $y' = y + dy$, whose difference to the preceding value y, i.e. dy, can be assigned by the usual rules of differentiation, will correspond to the following value $x' = x + dx$.

COROLLARY 3

§4 Since y is a function of x, $\frac{dy}{dx}$ will also be a function of x assignable by the given relation among x and y; and if one puts $\frac{dy}{dx} = p$, in like manner $\frac{dp}{dx}$ will be a certain function of x; but if we further set $\frac{dp}{dx} = q$, $\frac{dq}{dx} = r$, $\frac{dr}{dx} = s$ etc., even these quantities q, r, s etc. will be certain functions of x likewise assignable by the given relation among x and y.

COROLLARY 4

§5 Further, if V is an expression somehow composed of x and y, by the given relation among x and y it will also be of such a nature that it has determined values for all values of x. And if V' denotes the following value or the value corresponding to $x + dx$, it will be $V' = V + dV$ or $dV = V' - V$, according to the first principles of differential calculus.

HYPOTHESIS 2

§6 Whatever relation among x and y is propounded, since hence at the same time the relation among the differentials dy and dx is known, in the following I will always set:

$$\frac{dy}{dx} = p, \quad \frac{dp}{dx} = q, \quad \frac{dq}{dx} = r, \quad \frac{dr}{ds} = s \quad \text{etc.},$$

and p, q, r, s etc. will be functions assignable in terms of x and y.

COROLLARY 1

§7 As the letter p contains the relation of the differentials dx and dy, so q will contain the relation of the differentials of second order, r of the differentials of third order, s of fourth order etc.

COROLLARY 2

§8 Therefore, even vice versa, if there are differentials either of first order or of second order or even of higher order in the expression V, they can be thrown out by introducing these quantities p, q, r, s etc.

AXIOM

§9 If another relation among the variables x and y differing only infinitely less from the propounded one is constituted, the values of y corresponding to each value of x will also differ only infinitely less from those the propounded relation yields.

COROLLARY 1

§10 Since a varied relation of this kind can differ from the propounded relation in infinitely many ways so that the difference is infinitely small, it can happen that one or more values of y corresponding to certain values of x do not undergo a change.

COROLLARY 2

§11 This so general variation of the relation can be understood that hence all values of y undergo some changes not depending on each other. Therefore, to be as general as possible, it will be convenient to understand the variation of the conceived relation in this most general sense.

HYPOTHESIS 3

§12 If the propounded relation among x and y is changed infinitely less, let us denote the value of y corresponding to x after this by $y + \delta y$, so that δy denotes the variation y undergoes because the relation is varied.

COROLLARY 1

§13 Since in like manner y' is the value corresponding to $x + dx$ via the propounded relation, let us express its value corresponding to $x + dx$ via the varied relation by $y' + \delta y'$, so that $\delta y'$ denotes the variation of y' resulting from the variation of the relation.

COROLLARY 2

§14 Therefore, since $y' = y + dy$, it will be

$$\delta y' = \delta(y + dy) = \delta y + \delta dy \quad \text{and} \quad \delta dy = \delta y' - \delta y.$$

But δdy will denote the variation of dy resulting from the relation among x and y.

COROLLARY 3

§15 But as y' denotes the following state of y, having related the following state to $x + dx$, of course, so $\delta y'$ denotes the following state of δy, whence $\delta y' - \delta y$ will express the differential of δy, which is $d\delta y$. Therefore, since $\delta dy = \delta y' - \delta y$, it will be $\delta dy = d\delta y$.

COROLLARY 4

§16 Therefore, hence we derive this extraordinary property: The variation of the differential of y is equal to the differential of the variation of y. For, δdy is the variation of dy, i.e. the differential of y, and $d\delta y$ is the differential of δy, i.e. the variation of y.

Definition 1

§17 If V is an expression somehow conflated of x and y, having propounded a relation among x and y, its variation, which I will indicate by δV, is the increment the quantity V receives, if the propounded relation among x and y is varied infinitely less.

Corollary 1

§18 Therefore, the differential dV is to be distinguished carefully from the variation δV; for, the differential denotes the increment of V, if x is increased by its element dx, while the propounded relation among x and y remains the same; but the variation denotes the increment of V, if the relation itself is varied while x remains the same.

Corollary 2

§19 Since from the variation of the propounded relation among x and y the quantity y receives the increment δy while x remains the same: No matter how the quantity V was conflated of x and y, its variation will be found, if one writes $y + \delta y$ instead of y everywhere and V is subtracted from the value of V to result from that substitution.

Corollary 3

§20 If one writes $y + \delta y$ instead of y in V everywhere, the varied value of V will result, which is $V + \delta V$; but the variation itself is found, if the primitive value V is subtracted from the varied value $V + \delta V$.

Definition 2

§21 The calculus of variations is the method to find variations quantities somehow conflated of the two variables x and y undergo, if the propounded relation among x and y is changed infinitely less in some way.

Corollary 1

§22 Therefore, having propounded a relation among x and y, if V denotes a quantity somehow depending on x and y, this calculus teaches how to find the variation of V or the value of δV.

Corollary 2

§23 Since we assume the given relation among x and y to be changed somehow that y for each value of x undergoes some variations, which do not depend on each other, this calculus extends very far and can also be accommodated to given conditions of the variations.

Scholium 1

§24 For the calculus to be understood more clearly let us give an example. Therefore, let this relation among x and y be propounded

$$aayy - bbxx = aabb,$$

which, writing $b + db$ instead of b, is changed infinitely less. If now a quantity depending on x and y is propounded, e.g.,

$$\int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}},$$

its variation to result from that change of the relation can be exhibited applying this calculus; for, since

$$y = \frac{b}{a}\sqrt{aa - xx},$$

it will be

$$\delta y = \frac{db}{a}\sqrt{aa - xx},$$

which is the variation of y. But how from the known variation of y the variations of quantities depending somehow on y and x and even on

$$\int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}},$$

have to be determined, is to be shown in this calculus; hence it is plain that everything what has been treated in several different places by several authors on the variability of the parameter is contained here. Furthermore, these questions can even be inverted, as if, e.g., having propounded a formula of this kind

$$\int \frac{\sqrt{dx^2+dy^2}}{\sqrt{y}},$$

the relation among x and y is in question, whence the variation of this given formula is of a given magnitude or even zero, in which second case the found relation will give the maximum or minimum value of the propounded formula; and indeed all problems, that have been considered on curves enjoying a property of a maximum or a minimum, are to be referred to this.

SCHOLIUM 2

§25 The prescriptions of this calculus are accommodated to the diversity of the nature, according to which the propounded formula V depends on the two variables x and y; since the amount of these diversities is infinite, it will be convenient to divide them into some certain classes or species. Therefore, the first class contains the formulas which are somehow composed of the quantities x and y and those derived from them, i.e.

$$p=\frac{dy}{dx}, \quad q=\frac{dp}{dx}, \quad r=\frac{dq}{dx}\text{etc.},$$

but nevertheless in such a way that they do not involve integral formulas. To the second class I refer the formulas containing integral formulas like $\int Zdx$, so that Z belongs to the first class. The third class will comprehend formulas, in which not only the integrals $\int Zdx$ are contained, but also the quantity Z involves integrals. Finally, the fourth class follows, in which the formula V to be varied is not defined absolutely but just by a differential equation either of first order or of second order or higher order, which class extends very far and contains all the others as a special case. But concerning the equation expressing the relation among x and y, even though I consider it to be given, I nevertheless not define it to not restrict the rules to be be derived in the following somehow.

THEOREM 1

§26 The variation of a certain quantity V is equal to the differential of the variation of the same quantity or, in other words, $\delta dV = d\delta V$.

PROOF

Since $dV = V' - V$, while V' denotes the following value of V, i.e. the value corresponding to $x + dx$, and V corresponds to x, it will be $\delta dV = \delta V' - \delta V$; but $d\delta V$ expresses the difference of δV and its following value, i.e. $\delta V'$, so that $d\delta V = \delta V' - \delta V$, whence it is perspicuous that $\delta dV = d\delta V$.

COROLLARY 1

§27 The same way, if we write dV instead of V, it is plain that $\delta ddV = d\delta dV$; but $\delta dV = d\delta V$, whence $d\delta dV = dd\delta V$, and hence these three formulas will be equal

$$\delta dV = d\delta dV = d\delta V.$$

COROLLARY 2

§28 Further, if we write dV instead of V in that last formula, we will obtain an equality among these four formulas

$$\delta dddV = d\delta ddV = dd\delta dV = ddd\delta V,$$

but then among these five

$$\delta d^4 V = d\delta d^3 V = d^2\delta d^2 V = d^3\delta dV = d^4\delta V.$$

COROLLARY 3

§29 If one has the differential of any order of V, i.e. $d^n V$, whose variation is to be investigated, it will be

$$\delta d^n V = d^m \delta d^{n-m} V = d^n \delta V,$$

of course, it will be equal to the differential of n-th order of the variation δV. Therefore, hence the variation of differentials is reduced to the differentiation of the variation.

PROBLEM 1

§30 *To determine the variations of the quantities p, q, r, s etc. containing the ratio of the differentials of x and y.*

SOLUTION

Since the variation does not extend to x, it will be $\delta x = 0$ and the variation of y, i.e. δy, is considered to be known. Hence, since $p = \frac{dy}{dx}$, it will be

$$\delta p = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}.$$

Further, since $q = \frac{dp}{dx}$, it will be

$$\delta q = \frac{\delta dp}{dx} = \frac{d\delta p}{dx};$$

but for constant element dx we have $d\delta p = \frac{dd\delta y}{dx}$ and hence

$$\delta q = \frac{dd\delta y}{dx^2} \quad \text{and} \quad d\delta q = \frac{d^3\delta y}{dx^3}.$$

But furthermore, since $r = \frac{dq}{dx}$, it will be

$$\delta r = \frac{\delta dq}{dx} = \frac{d\delta q}{dx} \quad \text{and hence} \quad \delta r = \frac{d^3\delta y}{dx^3},$$

whence the variations of the quantities derived from x and y, i.e. p, q, r, s etc., will look as follows

$$\delta p = \frac{d\delta y}{dx}, \quad \delta q = \frac{d^2\delta y}{dx^2}, \quad \delta r = \frac{d^3\delta y}{dx^3}, \quad \delta s = \frac{d^4\delta y}{dx^4}\text{etc.},$$

if the element dx is assumed to be constant, of course.

COROLLARY 1

§31 These differentials of first and higher orders of the variation δy are determined by the variations of the values of y corresponding to the following values of x, i.e. $x + dx$, $x + 2x$, $x + 3dx$ etc. For, if the following values of y are exhibited this way: y', y'', y''', y'''' etc. and their variations this way: $\delta y'$, $\delta y''$, $\delta y'''$, $\delta y''''$, we know from the nature of differentials that

$$\begin{aligned} d\delta y &= \delta y' - \delta y, \\ dd\delta y &= \delta y'' - 2\delta y' + \delta y, \\ d^3\delta y &= \delta y''' - 3\delta y'' + 3\delta y' - \delta y \end{aligned}$$
$$\text{etc.}$$

COROLLARY 2

§32 Therefore, if only the value y is subjected to the variation, but the following ones y', y'', y''' etc. are not, that $\delta y' = 0$, $\delta y'' = 0$, $\delta y''' = 0$ etc., it will be

$$d\delta y = -\delta y, \quad dd\delta y = +\delta y, \quad d^3\delta y = -\delta y,, \quad d^4\delta y = +\delta y \quad \text{etc.}$$

and hence

$$\delta p = -\frac{\delta y}{dx}, \quad \delta q = +\frac{\delta y}{dx^2}, \quad \delta r = -\frac{\delta y}{dx^3}, \quad \delta s = +\frac{\delta y}{dx^4} \quad \text{etc.}$$

PROBLEM 2

§33 *If V was a quantity somehow conflated of the variables x and y and their differentials of any order or if it was an arbitrary function of the quantities x, y, p, q, r, s etc., to determine its variation δV.*

SOLUTION

Differentiate the function V as usual and let this expression result

$$dV = Mdx + Ndy + Pdp + Qdq + Pdr + Sds + \text{etc.},$$

which differential is nothing else but the increment the function V receives, if the quantities $x + dx$, $y + dy$, $p + dp$, $q + dq$, $r + dr$ etc. are substituted for x, y, p, q, s etc. In like manner, if for x, y, p, q, s etc. these are substituted

$$x + 0, \quad y + \delta y, \quad p + \delta p, \quad q + \delta q, \quad r + \delta r, \quad s + \delta s \quad \text{etc.},$$

the increment the function V receives from this will be its variation

$$\delta V = N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.}$$

Hence, if the values found above are for δp, δq, δr etc. are written here, the variation in question will result as

$$\delta V = N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} + \text{etc.}$$

THEOREM 2

§34 Having propounded an arbitrary integral formula Zdx, its variation will be equal to the integral of the variation of the differential Zdx or it will be

$$\delta \int Zdx = \int \delta Zdx.$$

PROOF

Since $\int Zdx$ expresses the sum of all Zdx, its variation $\delta \int Zdx$ will comprehend the sum of all variations of Zdx or it will be $\delta \int Zdx = \int \delta Zdx$. This can also shown more detailed this way: Let $\int Zdx = V$ so that one has to determine δV; therefore, since $dV = Zdx$, it will be $\delta dV = \delta Zdx = d\delta V$, whence, having taken the integrals, it will be $\delta V = \int \delta Zdx$.

PROBLEM 3

§35 *Having propounded the integral formula $\int Zdx$, in which Z is a quantity somehow conflated of x and y and their differentials of any order, to investigate its variation $\delta \int Zdx$.*

SOLUTION

Therefore, since Z is a function of x, y, p, q, r, s etc., its differential taken as usual will have a form of this kind

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.},$$

whence the variation of the same quantity Z will be

$$\delta Z = N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} + \text{etc.}$$

Since $\delta \int Zdx = \int \delta Zdx$, it will be

$$\delta \int Zdx = \int N\delta ydx + \int Pd\delta y + \int \frac{Qdd\delta y}{dx} + \int \frac{Rd^3\delta y}{dx^2} + \text{etc.};$$

for the expression δy not to disturb in the further reduction, let us put $\delta y = w$, and the reductions will look as follows

$$\begin{aligned}
\int Pdw &= Pw - \int wdP, \\
\int \frac{Qddw}{dx} &= \frac{Qdw}{dx} - \int \frac{dQ}{dx}dw = \frac{Qdw}{dx} - \frac{wdQ}{dx} + \int \frac{wddQ}{dx}, \\
\int \frac{Rd^3w}{dx^2} &= \frac{Rddw}{dx^2} - \frac{dRd\omega}{dx^2} + \frac{wddR}{dx^2} - \int \frac{wd^3R}{dx^2}
\end{aligned}$$

etc.

Collect all these values and substitute δy for w again, and this way one will obtain

$$\begin{aligned}
\delta \int Zdx = \int \delta y dx & \left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right) \\
&+ \delta y \left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right) \\
&+ \frac{d\delta y}{dx} \left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right) \\
&+ \frac{dd\delta y}{dx^2} \left(R - \frac{dS}{dx} + \text{etc.}\right) \\
&+ \frac{d^3\delta y}{dx^3} \left(S - \text{etc.}\right) \\
&+ \text{etc.},
\end{aligned}$$

in which expression the differential dx was assumed to be constant.

COROLLARY 1

§36 Therefore, the variation of the integral formula $\int Zdx$ consists of the integral part

$$\int \delta y dx \left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right)$$

and absolute parts, which except for the variation δy also contain its differentials $d\delta y$, $dd\delta y$, $d^3\delta y$ etc.

Corollary 2

§37 But we set up the integral part by the used reductions in such a way that it only contains the variation δy and is exhibited without its differentials, which form is of greatest use in the application of the calculus of variations.

Problem 4

§38 *If in the integral formula $\int Zdx$ the quantity Z does not only contain the letters x and y with the relations of the differentials p, q, r, s etc., but also contains the integral formula $\Pi = \int \mathfrak{Z}dx$ somehow, in which $\mathfrak{Z}$ is a function of x, y, p, q, r, s etc., to define the variation of the integral formula $\int Zdx$.*

Solution

Since the quantity Z except for the quantities x, y, p, q, r, s etc. also involves the integral formula $\Pi = \int \mathfrak{Z}dx$, it can be considered as a function of the quantities Π, x, y, p, q, r, s etc., whence, if it is differentiated in usual manner, this form will result

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + Sds\text{etc.},$$

whence the variation of Z is concluded to be

$$\delta Z = L\delta\Pi + N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.}$$

Further, since $\mathfrak{Z}$ is a function of x, y, p, q, r, s etc., put

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \mathfrak{S}ds + \text{etc.},$$

and from the preceding problem $\delta\Pi$ will be

$$\begin{aligned}\delta\int \mathfrak{Z}dx = \int \delta y dx \left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\right) \\ + \delta y \left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \frac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right) \\ + \frac{d\delta y}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \frac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right) \\ + \frac{dd\delta y}{dx^2}\left(\mathfrak{R} - \frac{d\mathfrak{S}}{dx} + \text{etc.}\right) \\ + \frac{d^3\delta y}{dx^3}\left(\mathfrak{S} - \text{etc.}\right) \\ + \text{etc.}\end{aligned}$$

Or rather take the first form

$$\delta\int \mathfrak{Z}dx = \int \mathfrak{N}\delta y dx + \int \mathfrak{P} d\delta y + \int \frac{\mathfrak{Q} dd\delta y}{dx} + \int \frac{\mathfrak{R} d^3\delta y}{dx^2} + \int \frac{\mathfrak{S} d^4\delta y}{dx^3} + \text{etc.},$$

and, because of $\delta\Pi = \delta\int \mathfrak{Z}dx$, it will be

$$\begin{aligned}\delta Z = L\int \mathfrak{N}\delta y dx + L\int \mathfrak{P} d\delta y + L\int \frac{\mathfrak{Q} dd\delta y}{dx} + L\int \frac{\mathfrak{R} d^3\delta y}{dx^2} + L\int \frac{\mathfrak{S} d^4\delta y}{dx^3} + \text{etc.} \\ + N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} + \text{etc.}\end{aligned}$$

Therefore, since $\delta\int Zdx = \int \delta Z dx$, we will have

$$\begin{aligned}\delta\int Zdx = \int Ldx \int \mathfrak{N}\delta y dx + \int Ldx \int \mathfrak{P} d\delta y \\ + \int Ldx \int \frac{\mathfrak{Q} dd\delta y}{dx} + \int Ldx \int \frac{\mathfrak{R} d^3\delta y}{dx^2} + \text{etc.} \\ + \int N\delta y dx + \int P d\delta y + \int \frac{Qdd\delta y}{dx} + \int \frac{Rd^3\delta y}{dx^2} + \text{etc.}\end{aligned}$$

Put $\int Ldx = W$, or $Ldx = dW$, and, because of

$$\begin{aligned}\int Ldx \int \mathfrak{N}\delta y dx &= W\int \mathfrak{N}\delta y \delta x - \int \mathfrak{N} W \delta y dx, \\ \int Ldx \int \mathfrak{P} d\delta y &= W\int \mathfrak{P} d\delta y - \int \mathfrak{P} W d\delta y, \\ \int Ldx \int \frac{\mathfrak{Q} dd\delta y}{dx} &= W\int \frac{\mathfrak{Q} dd\delta y}{dx} - \int \frac{\mathfrak{Q} W dd\delta y}{dx},\end{aligned}$$

we will obtain:

$$\begin{aligned}\delta\int Zdx = W\int \mathfrak{N}\delta y dx + W\int \mathfrak{P}d\delta y + W\int \frac{\mathfrak{Q}dd\delta y}{dx} + W\int \frac{\mathfrak{N}d^3\delta y}{dx^2} + \text{etc.}\\ + \int (N-\mathfrak{N}W)\delta y dx + \int (P-\mathfrak{P}W)d\delta y\\ + \int (Q-\mathfrak{Q}W)\frac{dd\delta y}{dx} + \int (R-\mathfrak{R}W)\frac{d^3\delta y}{dx^2} + \text{etc.}\end{aligned}$$

These formulas, reduced the same way as above, will give

$$\begin{aligned}&\delta\int Zdx\\ &= W\int \delta y dx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \text{etc.}\right)\\ &+ W\delta y\left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \text{etc.}\right)\\ &+ \frac{Wd\delta y}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \text{etc.}\right)\\ &+ \frac{Wdd\delta y}{dx^2}\left(\mathfrak{R} - \text{etc.}\right)\\ &+ \int \delta y dx\left((N-\mathfrak{N}W) - \frac{d(P-\mathfrak{P}W)}{dx} + \frac{dd(Q-\mathfrak{Q}W)}{dx^2} - \frac{d^3(R-\mathfrak{R}W)}{dx^3} + \text{etc.}\right)\\ &+ \delta y\left((P-\mathfrak{P}W) - \frac{d(Q-\mathfrak{Q}W)}{dx} + \frac{dd(R-\mathfrak{R}W)}{dx^2} - \text{etc.}\right)\\ &+ \frac{d\delta y}{dx}\left((Q-\mathfrak{Q}W) - \frac{d(R-\mathfrak{R}W)}{dx} + \text{etc.}\right)\\ &+ \frac{dd\delta y}{dx^2}\left((R-\mathfrak{R}W) - \text{etc.}\right)\\ &+ \text{etc.}\end{aligned}$$

COROLLARY 1

§39 Since the applied reductions can easily be done in each case, having mentioned them in advance, having put $W = \int Ldx$, the variation in question can exhibited more succinctly this way

$$\delta\int Zdx = W\int dx\left(\mathfrak{N}\delta y + \frac{\mathfrak{P}d\delta y}{dx} + \frac{\mathfrak{Q}dd\delta y}{dx^2} + \frac{\mathfrak{R}d^3\delta y}{dx^3} + \text{etc.}\right)$$
$$+\int dx\left((N-\mathfrak{N}W)\delta y + (P-\mathfrak{P}W)\frac{d\delta y}{dx}\right.$$
$$\left.+(Q-\mathfrak{Q}W)\frac{dd\delta y}{dx^2} + (R-\mathfrak{R}W)\frac{d^3\delta y}{dx^3} + \text{etc.}\right).$$

COROLLARY 2

§40 And if the quantity Z additionally involves another integral formula $\Pi' = \int \mathfrak{Z}dx$, so that

$$dZ = Ld\Pi + L'd\Pi' + Mdx + Ndy + Pdp + Qdq + \text{etc.},$$

but then

$$d\mathfrak{Z}' = \mathfrak{M}'dx + \mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr + \text{etc.},$$

if one puts $\int Ldx = W$, $\int L'dx = W'$ and additionally, for the sake of brevity:

$$N - \mathfrak{N}W - \mathfrak{N}'W' = (N);\quad P - \mathfrak{P}W - \mathfrak{P}'W' = (P),$$
$$Q - \mathfrak{Q}W - \mathfrak{Q}'W' = (Q);\quad R - \mathfrak{R}W - \mathfrak{R}'W' = (R)\text{etc.},$$

the variation in question will be:

$$\delta\int Zdx = W\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$
$$+W'\int dx\left(\mathfrak{N}'\delta y + \mathfrak{P}'\frac{d\delta y}{dx} + \mathfrak{Q}'\frac{dd\delta y}{dx^2} + \mathfrak{R}'\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$
$$+\int dx\left((N)\delta y + (P)\frac{d\delta y}{dx} + (Q)\frac{dd\delta y}{dx^2} + (R)\frac{d^3\delta y}{dx^3} + \text{etc.}\right).$$

PROBLEM 5

§41 *If in the formula $\int Zdx$ the quantity Z, aside from the letters x, y, p, q, r etc. also involves the integral formula $\Pi = \int \mathfrak{Z}dx$, in which the quantity $\mathfrak{Z}$, aside from the letters x, y, p, q, r etc., additionally contains the integral formula $\pi = \int \mathfrak{z}dx$, where $\mathfrak{z}$ is a function only of the letters x, y, p, q, r etc., to find the variation of the formula $\int Zdx$.*

SOLUTION

Since Z is a function of the quantities x, y, p, q, r, s etc. and $\Pi = \int \mathfrak{Z}dx$, its differential taken in usual manner will be of this form

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

and hence the variation

$$\delta Z = L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + \text{etc.},$$

whence the variation in question will be

$$\delta \int Zdx = \int \delta Zdx$$
$$= \int Ldx\delta\Pi + \int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + \text{etc.}\right).$$

But since $\mathfrak{Z}$ is a function of the quantities x, y, p, q, r etc. and $\pi = \int \mathfrak{z}dx$, by differentiation it will be

$$d\mathfrak{Z} = \mathfrak{L}d\pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

and hence its variation

$$\delta\mathfrak{Z} = \mathfrak{L}\delta\pi + \mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.},$$

whence, since $\Pi = \int \mathfrak{Z}dx$, it will be $\delta\Pi = \delta \int \mathfrak{Z}dx = \int \delta\mathfrak{Z}dx$ and therefore

$$\delta\Pi = \int \mathfrak{L}dx\delta\pi + \int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right),$$

whence one finds

$$\int Ldx\delta\Pi = \int Ldx \int \mathfrak{L}dx\delta\pi + \int Ldx \int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right).$$

Therefore, it remains to define $\delta\pi$; but $\pi = \int \mathfrak{z}dx$, and since $\mathfrak{z}$ is a function of the letters x, y, p, q, r, s etc., by differentiation let:

$$d\mathfrak{z} = \mathfrak{m}dx + \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + \text{etc.},$$

whence the variation is concluded to be:

$$\delta\mathfrak{z} = \mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.},$$

but then, because of $\delta\pi = \delta \int \mathfrak{z}dx = \int \delta\mathfrak{z}dx$, it will be

$$\delta\pi = \int dx \left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.} \right).$$

Therefore, we will have

$$\int Ldx \int \mathfrak{L}dx\delta\pi = \int Ldx \int \mathfrak{L}dx \int dx \left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.} \right).$$

Now to liberate this formula from the multiple integral signs, let us put $\int Ldx = W$ and it will be:

$$\int Ldx\delta\Pi = W\delta\Pi - \int Wd\delta\Pi,$$

but $d\delta\Pi = \delta\mathfrak{Z}dx$, whence

$$\int Ldx\delta\Pi = W\delta\Pi - \int W\delta\mathfrak{Z}dx$$

and hence:

$$\begin{aligned} \int Ldx\delta\Pi = W \int \mathfrak{L}dx\delta\pi + W \int dx \left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.} \right) \\ - \int \mathfrak{L}Wdx\delta\pi \; - \int Wdx \left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.} \right). \end{aligned}$$

Let $\int \mathfrak{L}dx = \mathfrak{W}$, it will be

$$\int \mathfrak{L}dx\delta\pi = \mathfrak{W}\delta\pi - \int \mathfrak{W}\delta\mathfrak{z}dx$$

and hence:

$$\begin{aligned} \int \mathfrak{L}dx\delta\pi = \mathfrak{W} \int dx \left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.} \right) \\ - \int \mathfrak{W}dx \left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.} \right). \end{aligned}$$

Further, put $\int \mathfrak{L}Wdx = \int Wd\mathfrak{W} = \mathfrak{V}$ so that:

$$\int \mathfrak{L}Wdx\delta\pi = \mathfrak{V}\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \text{etc.}\right)$$
$$- \int \mathfrak{V}dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \mathfrak{r}\frac{d^3\delta y}{dx^3} + \right).$$

From all these the variation in question, i.e. $\delta\int Zdx$, will be concluded to be:

$$= (W\mathfrak{W} - \mathfrak{V})\int dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$-W\int \mathfrak{W}dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$+\int \mathfrak{V}dx\left(\mathfrak{n}\delta y + \mathfrak{p}\frac{d\delta y}{dx} + \mathfrak{q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$+W\int dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$-\int Wdx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)$$
$$+\int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right).$$

COROLLARY 1

§42 If the variation of the formula $\int Zdx$ extended from $x = 0$ to the value $x = a$ is in question, take the integrals $W = \int Ldx$, $\mathfrak{M} = \int \mathfrak{L}dx$ and $\mathfrak{V} = \int Wd\mathfrak{M}$, so that they vanish for $x = 0$, but then for $x = a$ let $W = A$, $\mathfrak{M} = \mathfrak{A}$ and $\mathfrak{V} = \mathfrak{B}$, which values can be written in the found formula for the letters W, $\mathfrak{M}$ and $\mathfrak{V}$, where they appear not under the integral sign.

COROLLARY 2

§43 Therefore, for the sake of brevity put:

$$\begin{aligned}
N + (A - W)\mathfrak{N} + (A\mathfrak{A} - \mathfrak{B} - A\mathfrak{W} + \mathfrak{V})\mathfrak{n} &= (N),\\
P + (A - W)\mathfrak{P} + (A\mathfrak{A} - \mathfrak{B} - A\mathfrak{W} + \mathfrak{V})\mathfrak{p} &= (P),\\
Q + (A - W)\mathfrak{Q} + (A\mathfrak{A} - \mathfrak{B} - A\mathfrak{W} + \mathfrak{V})\mathfrak{q} &= (Q),\\
R + (A - W)\mathfrak{R} + (A\mathfrak{A} - \mathfrak{B} - A\mathfrak{W} + \mathfrak{V})\mathfrak{r} &= (R)\\
&\text{etc.}
\end{aligned}$$

and the variation in question of the formula $\int Zdx$ extended to the value $x = a$ will be:

$$\int dx \left((N)\delta y + (P)\frac{d\delta y}{dx} + (Q)\frac{dd\delta y}{dx^2} + (R)\frac{d^3\delta y}{dx^3} + \text{etc.} \right).$$

COROLLARY 3

§44 But if the above reductions are used here, one will find the same variation expressed this way:

$$\begin{aligned}
\delta \int Zdx = \int dx\delta y &\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.} \right)\\
+ \ \delta y &\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.} \right)\\
+ \ \frac{d\delta y}{dx} &\left((Q) - \frac{d(R)}{dx} + \text{etc.} \right)\\
+ \ \frac{dd\delta y}{dx^2} &((R) - \text{etc.})\\
+ \ \text{etc.}
\end{aligned}$$

COROLLARY 4

§45 Since $\mathfrak{B} = \int Wd\mathfrak{W}$, it will be $A\mathfrak{W} - \mathfrak{V} = \int (A - W)\mathfrak{L}dx$; hence, if one puts the integral $\int (A - W)\mathfrak{L}dx = X$, taken in such a way that it vanishes for $x = 0$, but then becomes $X = B$ for $x = a$, so that:

$$\begin{aligned}
\int \mathfrak{L}dx = W, &\quad \text{and for} \quad x = a \quad \text{we have} \quad W = A,\\
\int (A - W)\mathfrak{L}dx = X, &\quad \text{and for} \quad x = a \quad \text{we have} \quad X = B \quad ,
\end{aligned}$$

the above values exhibited in corollary 2 will be as follows:

$$\begin{aligned}
N + (A-W)\mathfrak{N} + (B-X)\mathfrak{n} &= (N),\\
P + (A-W)\mathfrak{P} + (B-X)\mathfrak{p} &= (P),\\
Q + (A-W)\mathfrak{Q} + (B-X)\mathfrak{q} &= (Q),\\
R + (A-W)\mathfrak{R} + (B-X)\mathfrak{r} &= (R)\\
&\text{etc.}
\end{aligned}$$

Problem 6

§46 *If in the integral formula $\Phi = \int Zdx$ the quantity, aside from the letters x, y, p, q, r etc., also contains the integral formula Φ itself, to determine the variation $\delta\Phi = \delta\int Zdx$.*

Solution

Since Z is a function of the quantities x, y, p, q, r etc. and additionally involves the integral formula $\Phi = \int Zdx$, differentiate in usual manner and let this expression result

$$dZ = Ld\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

Therefore, the variation of Z will obviously be

$$\delta Z = L\delta\Phi + N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + \text{etc.},$$

and hence, because of $\delta\Phi = \delta\int Zdx = \int \delta Zdx$,

$$\delta\Phi = \int Ldx\delta\Phi + \int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + \text{etc.}\right).$$

Let us put $\delta\Phi = z$, since this is that itself what is in question, and for the sake of brevity also put

$$\int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right) = u,$$

so that one has $z = \int Lzdx + u$ and by differentiating $dz = Lzdx + du$, and it will be

$$z = e^{\int Ldx}\int e^{-\int Ldx}du;$$

for the sake of brevity set $\int Ldx = W$ and one has the variation in question

$$\delta \int Zdx = \mathrm{e}^{W} \int \mathrm{e}^{-W} dx \left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.} \right).$$

If the variation up to a given limit $x = a$ is desired and then $W = A$, for the sake of brevity put

$$\mathrm{e}^{A-W}N = (N), \quad \mathrm{e}^{A-W}P = (P), \quad \mathrm{e}^{A-W}Q = (Q) \quad \text{etc.},$$

and, by the reductions given above, it will be

$$\begin{aligned}\delta \int Zdx = \int dx\delta y & \left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.} \right) \\ + \quad \delta y & \left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.} \right) \\ + \quad \frac{d\delta y}{dx} & \left((Q) - \frac{d(R)}{dx} + \text{etc.} \right) \\ + \frac{dd\delta y}{dx^2} & \left((R) - \text{etc.} \right) \\ + \text{etc.} & \end{aligned}$$

COROLLARY

§47 Therefore, if the quantity Φ to be varied is defined by the differential equation $d\Phi = Zdx$, in which Z somehow involves the quantity Φ and additionally the letters x, y, p, q, r etc., its variation $\delta\Phi$ can be assigned by the result of this problem.

PROBLEM 7

§48 *If in the integral formula $\Phi = \int Zdx$ the quantity Z, aside from the letters x, y, p, q, r etc., does not only involve the quantity Φ but additionally even another integral formula $\Pi = \int \mathfrak{Z}dx$ somehow, in which the quantity $\mathfrak{Z}$ is only given by the letters x, y, p, q, r etc., to investigate the variation $\delta\Phi = \delta \int Zdx = \int \delta Zdx$.*

SOLUTION

Since Z is a function of the quantities x, p, q, r etc. and additionally of the quantities $\Phi = \int Zdx$ and $\Pi = \int \mathfrak{Z}dx$, let this expression result by differentiation

$$dZ = Kd\Phi + Ld\Pi + Mdx + Ndy + Pdp + Qdq + \text{etc.},$$

whence the variation will be

$$\delta Z = K\delta\Phi + L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}$$

Further, since $\mathfrak{Z}$ is a function of the letters x, y, p, q, r etc. only, put

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

and, because of $\delta\Pi = \int \delta\mathfrak{Z}dx$, it will be

$$\delta\Pi = \int dx \left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \mathfrak{R}\frac{d^3\delta y}{dx^3} + \text{etc.}\right).$$

As before set

$$\delta\Phi = z \quad \text{and} \quad L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.} = u;$$

because of $\delta\Phi = \int \delta Zdx = z$ it will be $\delta Z = \frac{dz}{dx}$ and hence $\frac{dz}{dx} = Kz + u$; therefore, it results

$$z = \mathrm{e}^{\int Kdx} \int \mathrm{e}^{-\int Kdx} udx = \delta\Phi;$$

let $\int Kdx = V$ and it will be

$$\begin{aligned}\mathrm{e}^{-\int Kdx}udx = \mathrm{e}^{-V}Ldx \int dx \left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right) \\ + \mathrm{e}^{-V}dx \left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right),\end{aligned}$$

further, put $\int e^{-V}Ldx = W$, and by integration the variation in question will be

$$\begin{aligned}\delta\Phi = \mathrm{e}^{V}W \int dx \left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right) \\ - \mathrm{e}^{V} \int Wdx \left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right) \\ + \mathrm{e}^{V} \int \mathrm{e}^{-V}dx \left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right).\end{aligned}$$

If the variation up to a given limit $x = a$ is desired and for $x = a$ we have $V = A$ and $W = B$, then, for the sake of brevity, set:

$$\begin{aligned} &\mathrm{e}^{A-V}N + \mathrm{e}^{A}(B-W)\mathfrak{N} = (N), \\ &\mathrm{e}^{A-V}P \ + \mathrm{e}^{A}(B-W)\mathfrak{P} = (P), \\ &\mathrm{e}^{A-V}Q + \mathrm{e}^{A}(B-W)\mathfrak{Q} = (Q), \\ &\mathrm{e}^{A-V}R \ + \mathrm{e}^{A}(B-W)\mathfrak{R} = (R) \\ &\qquad\qquad\qquad \text{etc.,} \end{aligned}$$

having done which the variation of the formula $\Phi = \int Zdx$ extended up to the limit $x = a$ is:

$$\begin{aligned} \delta\Phi = \int dx\delta y &\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.}\right) \\ + \quad \delta y \ &\left((P) \ - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.}\right) \\ + \ \frac{d\delta y}{dx} &\left((Q) - \frac{d(R)}{dx} + \text{etc.}\right) \\ + \ \frac{dd\delta y}{dx^2} &\big((R) \ \ - \text{etc.}\big) \end{aligned}$$

COROLLARY

§49 Therefore, this way the variation of the quantity Φ given by the differential equation $d\Phi = Zdx$ is defined, in which Z, aside from the letters x, y, p, q, r etc., does not only contain Φ itself, but also involves the integral formula $\int \mathfrak{Z}dx = \Pi$ somehow, as long as $\mathfrak{Z}$ is determined by the letters x, y, p, q, r etc. only.

PROBLEM 8

§50 *If in the integral formula $\Phi = \int Zdx$ the quantity Z, aside from the letters x, y, p, q, r etc., involves the integral formula $\Pi = \int \mathfrak{Z}dx$, but there the quantity $\mathfrak{Z}$, aside from the letters x, y, p, q, r etc., contains the integral formula $\Pi = \int \mathfrak{Z}dx$, to define the variation of the propounded formula $\Phi = \int Zdx$.*

Since Z is a function of the quantities x, y, p, q, r and of $\Pi = \int \mathfrak{Z} dx$, its differential will be of this form:

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

hence the variation will be

$$\delta Z = L\delta\Pi + N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{d^3\delta y}{dx^3} + \text{etc.},$$

whence, because of $\delta\Phi = \int \delta Z dx$, one will have:

$$\delta\Phi = \int Ldx\delta\Pi + \int dx\left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right).$$

But since $\mathfrak{Z}$ is a function of x, y, p, q, r etc. and $\Pi = \int \mathfrak{Z} dx$, let its differential be:

$$d\mathfrak{Z} = \mathfrak{L}d\Pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \text{etc.},$$

and it will be

$$d\mathfrak{Z} = \frac{d\delta\Pi}{dx} = \mathfrak{L}\delta\Pi + \mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}$$

Put $\int \mathfrak{L}dx = \mathfrak{W}$, and it will be:

$$\delta\Pi = \mathrm{e}^{\mathfrak{W}}\int \mathrm{e}^{-\mathfrak{W}}dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right).$$

Let $\int \mathrm{e}^{\mathfrak{W}}Ldx = W$ and one will obtain:

$$\begin{aligned}\delta\Phi = W&\int \mathrm{e}^{-\mathfrak{W}}dx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)\\ &- \int \mathrm{e}^{-\mathfrak{W}}Wdx\left(\mathfrak{N}\delta y + \mathfrak{P}\frac{d\delta y}{dx} + \mathfrak{Q}\frac{dd\delta y}{dx^2} + \text{etc.}\right)\\ &+ \int dx\left(Ndy + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + \text{etc.}\right).\end{aligned}$$

If this variation is to be extended up to the limit $x = a$ and for $x = a$ we have $W = A$, for the sake of brevity call

$$\begin{aligned}N + \mathrm{e}^{-\mathfrak{W}}(A - W)\mathfrak{N} &= (N),\\ P + \mathrm{e}^{-\mathfrak{W}}(A - W)\mathfrak{P} &= (P),\\ Q + \mathrm{e}^{-\mathfrak{W}}(A - W)\mathfrak{Q} &= (Q)\end{aligned}$$

etc.,

and, introducing the reductions given above, the variation of the integral formula $\Phi = \int Zdx$ extended up to the limit $x = a$ will be

$$\begin{aligned}
\delta \int Zdx = \int dx\delta y & \left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \text{etc.} \right) \\
+ \quad \delta y & \left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \text{etc.} \right) \\
+ \quad \frac{d\delta y}{dx} & \left((Q) - \frac{d(R)}{dx} + \text{etc.} \right) \\
+ \frac{dd\delta y}{dx^2} & \left((R) - \text{etc.} \right)
\end{aligned}$$

etc.

SCHOLIUM

§51 The use of this problem will be seen considering bodies descending along curves in an arbitrary resisting medium, while the body is acted upon by arbitrary forces, if we want to define the variation of the time of the descent, while the curve is varied arbitrarily. In this case let Φ denote the time of the descent along the arc corresponding to the abscissa x and let the ordinate be y and Π the altitude due to the acquired velocity; and the time of descent will be

$$\Phi = \int \frac{dx\sqrt{1+pp}}{\sqrt{\Pi}},$$

having put $dy = pdx$, so that $dx\sqrt{1+pp}$ denotes the line element. But from the action on the body it will be

$$d\Pi = Xdx + Ydy - V\sqrt{dx^2 + dy^2},$$

where X and Y denotes functions of x and y, and V a function of Π proportional to the resistance. Therefore, because of $dy = pdx$, it will be

$$\Pi = \int (X + Yp - V\sqrt{1+pp})dx$$

and hence

$$\mathfrak{Z} = X + Yp - V\sqrt{1+pp},$$

while $Z = \frac{\sqrt{1+pp}}{\sqrt{\Pi}}$.

COROLLARY

§52 If, for the sake of uniformity, one puts

$$M + \mathrm{e}^{-\mathfrak{W}}(A - W)\mathfrak{W} = (M),$$

$(M)dx + (N)dy + (P)dp + (Q)dq + (R)dr +$ etc. will be the true differential of this formula:

$$Z + \mathrm{e}^{-\mathfrak{W}}(A - W)\mathfrak{Z}.$$

CONCLUSION

§53 Therefore, whatever integral formula $\Phi = \int Zdx$ is propounded, whose variation is to be investigated, its variation extended up to the limit $x = a$ will be expressed this way

$$\begin{aligned}\delta\Phi = \int dx\delta y &\left((N) - \frac{d(P)}{dx} + \frac{dd(Q)}{dx^2} - \frac{d^3(R)}{dx^3} + \frac{d^4(S)}{dx^4} - \text{etc.}\right)\\ + \quad \delta y \quad &\left((P) - \frac{d(Q)}{dx} + \frac{dd(R)}{dx^2} - \frac{d^3(S)}{dx^3} + \text{etc.}\right)\\ + \quad \frac{d\delta y}{dx} &\left((Q) - \frac{d(R)}{dx} + \frac{dd(S)}{dx^2} - \text{etc.}\right)\\ + \frac{dd\delta y}{dx^2} &\left((R) - \frac{d(S)}{dx} + \text{etc.}\right)\\ + \frac{d^3\delta y}{dx^3} &\left((S) - \text{etc.}\right)\\ + \text{etc.},\end{aligned}$$

having assumed the element dx to be constant. But how the letters (N), (P), (Q), (R), (S) etc. look, will become clear in each case.

CASE I

§54 If $dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds +$ etc., it will be

$$(N) = N, \quad (P) = P, \quad (Q) = Q, \quad (R) = R, \quad (S) = S \quad \text{etc.}$$

Case II

§55 If $dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$, while $\Pi = \int \mathfrak{Z}dx$ and

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

let $\int Ldx = W$ and for $x = a$ let $W = A$, whence:

$$\begin{array}{ll} (N) = N + (A - W)\mathfrak{N} & (P) = P + (A - W)\mathfrak{P} \\ (Q) = Q + (A - W)\mathfrak{Q} & (R) = R + (A - W)\mathfrak{R} \\ (S) = S + (A - W)\mathfrak{S} & \quad\text{etc.} \end{array}$$

Case III

§56 If it was

$$dZ = Ld\Pi + L'd\Pi' + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

while $\Pi = \int \mathfrak{Z}dx$ and $\Pi' = \int \mathfrak{Z}'dx$, but then

$$\begin{aligned} d\mathfrak{Z} &= \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}, \\ d\mathfrak{Z}' &= \mathfrak{M}'dx + \mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr + \text{etc.}, \end{aligned}$$

put $\int Ldx = W$ and $\int L'dx = W'$, and for $x = a$ let $W = A$ and $W' = A'$, having done which it will be:

$$\begin{aligned} (N) &= N + (A - W)\mathfrak{N} + (A' - W')\mathfrak{N}' \\ (P) &= P + (A - W)\mathfrak{P} + (A' - W')\mathfrak{P}' \\ (Q) &= Q + (A - W)\mathfrak{Q} + (A' - W')\mathfrak{Q}' \\ (R) &= R + (A - W)\mathfrak{R} + (A' - W')\mathfrak{R}' \\ &\text{etc.} \end{aligned}$$

Case IV

§57 If Z contains the integral formula $\Pi = \int \mathfrak{Z}dx$, so that

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

the quantity $\mathfrak{Z}$ on the other hand contains $\pi = \int \mathfrak{z}dx$ that:

$$d\mathfrak{Z} = \mathfrak{L}d\pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

but $\mathfrak{z}$ does not involve any further integral so that:

$$d\mathfrak{z} = \mathfrak{m}dx + \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + \text{etc.},$$

put $\int Ldx = W$ and for $x = a$ let $W = A$; but then put $\int (A - W)\mathfrak{L}dx = \mathfrak{W}$ and for $x = a$ let $\mathfrak{W} = \mathfrak{A}$, having done which it will be:

$$\begin{aligned}(N) &= N + (A - W)\mathfrak{N} + (\mathfrak{A} - \mathfrak{W})\mathfrak{n},\\ (P) &= P + (A - W)\mathfrak{P} + (\mathfrak{A} - \mathfrak{W})\mathfrak{p},\\ (Q) &= Q + (A - W)\mathfrak{Q} + (\mathfrak{A} - \mathfrak{W})\mathfrak{q},\\ (R) &= N + (A - W)\mathfrak{R} + (\mathfrak{A} - \mathfrak{W})\mathfrak{r}\\ &\text{etc.}\end{aligned}$$

Case V

§58 Since Z contains the formula $\Phi = \int Zdx$ itself so that

$$dZ = Kd\Phi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

put $\int Kdx = V$ and for $x = a$ let $V = C$; it will be:

$$(N) = \mathrm{e}^{C-V}N, \quad (P) = \mathrm{e}^{C-V}P, \quad (Q) = \mathrm{e}^{C-V}Q, \quad (R) = \mathrm{e}^{C-V}R \quad \text{etc.}$$

Case VI

§59 If Z, aside from the formula $\Phi = \int Zdx$, contains another integral formula $\Pi = \int \mathfrak{Z}dx$ and:

$$dZ = Kd\Phi + Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

but $\mathfrak{Z}$ does not contain another integral formula:

$$d\mathfrak{Z} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

let $\int Kdx = V$ and for $x = a$ let $V = C$. Furthermore, let $\int \mathrm{e}^{C-V}Ldx = W$ and for $x = a$ let $W = A$, and it will be

$$\begin{aligned}(N) &= \mathrm{e}^{C-V}N + (A - W)\mathfrak{N},\\ (P) &= \mathrm{e}^{C-V}P + (A - W)\mathfrak{P},\\ (Q) &= \mathrm{e}^{C-V}Q + (A - W)\mathfrak{Q},\\ (R) &= \mathrm{e}^{C-V}R + (A - W)\mathfrak{R}\\ &\text{etc.}\end{aligned}$$

CASE VII

§60 If Z contains the integral formula $\Pi = \int \mathfrak{Z} dx$ that:

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

but then $\mathfrak{Z}$ involves the same formula $\Pi = \int \mathfrak{Z} dx$ again that:

$$d\mathfrak{Z} = \mathfrak{L}d\Pi + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

put $\int \mathfrak{L}dx = \mathfrak{W}$ and for $x = a$ put $\mathfrak{W} = \mathfrak{A}$; furthermore, let

$$\int \mathrm{e}^{-\mathfrak{A}+\mathfrak{W}} L dx = W$$

and for $x = a$ let $W = A$, and it will be:

$$\begin{aligned} (N) &= N - \mathrm{e}^{\mathfrak{A}-\mathfrak{W}}(A - W)\mathfrak{N}, \\ (P) &= P - \mathrm{e}^{\mathfrak{A}-\mathfrak{W}}(A - W)\mathfrak{P}, \\ (Q) &= Q - \mathrm{e}^{\mathfrak{A}-\mathfrak{W}}(A - W)\mathfrak{Q}, \\ (R) &= R - \mathrm{e}^{\mathfrak{A}-\mathfrak{W}}(A - W)\mathfrak{R} \\ &\text{etc.} \end{aligned}$$

§61 In like manner this investigation can be extended to other complicated formulas, but since those do usually not occur, the effort would be superfluous. Therefore, since I taught how to the define the variations both of simple and more complicated integral formulas, the calculus of variations seems to be worked out almost completely; for, of whatever nature the quantity to be varied was, no matter whether conflated of absolute formulas or integral formulas, its variation can be found by ordinary differentiation. As if, e.g., the quantity to varied U contains some integral formulas like

$$\Phi = \int Zdx, \quad \Phi' = \int Z'dx, \quad \Phi'' = \int Z''dx \quad \text{etc.},$$

differentiate it in usual manner an let this expression result:

$$\delta U = Kd\Phi + K'd\Phi' + K''d\Phi'' \quad \text{etc.},$$

then, it is evident that its variation will be:

$$\delta U = K\delta\Phi + K'\delta\Phi' + K''\delta\Phi'' + \text{etc.},$$

but the variations $\delta\Phi$, $\delta\Phi'$, $\delta\Phi''$ etc. will be assigned using the prescriptions just explained. But at the same time it is plain that the variation δU will always be expressed in a form of such a kind that

$$\delta V = \int (A) dx \delta y + (B)\delta y + (C)\frac{d\delta y}{dx} + (D)\frac{dd\delta y}{dx^2} + \text{etc.},$$

where (A), (B), (C) etc. are functions to be found from the rules given above. But it will be convenient to display the utility of this calculus of variations briefly in the solution of the famous isoperimetric problem understood in the broadest sense.

Application of the Calculus of variations to the solution of the isoperimetric problem understood in the broadest sense

§62 The primary problem extending to this can be formulated in such a way that among all curves to be constructed over the same given base $x = a$ the one is defined, for which a certain formula U has a maximum or minimum value. For, even though the formulation of the problem only contains the length of the curve, this condition is nevertheless conveniently omitted for the problem to extend further and even the mentioning of the single formula U, whose value has to become maximal or minimal, is not to be considered to restrict it, after I had demonstrated in general: If among all curves to be constructed over the same base $x = a$, for which curves the formula V has the same value, the one curve must be defined, in which the value of the formula U becomes maximal or minimal, the question is reduced to this that among completely all curves to be constructed over the base $x = a$ the one is defined, for which this composited formula $\alpha V + \beta U$ has the maximum or minimum value. Nevertheless, even the reason for this reduction can be explained nicely from the principles of this calculus of variations.

§63 But this question can be propounded this way abstracted from the consideration of curved lines:

Having propounded an arbitrary formula U, to define the relation among two variables x and y, if by which the value of U is determined and it is extended from $x = 0$ to $x = a$, so that the maximum or minimum value for that U results.

Therefore, let us consider the relation among x and y as found already, so that hence the maximum or minimum value of U results, and it is obvious, if the relation among x and y is varied just a little, that hence no variation on the value of U results; or, what is the same, the variation of U or δU must become equal to zero; and so the equation $\delta U = 0$ contains the relation among x and y in question.

§64 But we taught how to define the variation δU by assuming that for each value of x the value of y, which corresponds to that x via the given relation, is increased by δy. Therefore, since the relation in question among all possible ones must enjoy these prerogatives, the variation δU must always be equal to zero, no matter how the values of y are increased by those particular δy and no matter of what nature these increasings were, since they are completely arbitrary and do not depend on each other. And it is not even necessary to attribute variations to all values of y, but no matter whether either just a single or two or arbitrarily many values are varied, it is always necessary that the variation, which hence follows for the complete value of U, if it is extended from $x = 0$ to $x = a$, becomes equal to zero.

§65 But from the results derived above it is obvious that the variation of U is always expressed as this:

$$\delta U = \int (A) dx \delta y + (B) \delta y + (C) \frac{d\delta y}{dx} + (D) \frac{dd\delta y}{dx^2} + \text{etc.},$$

each part of which form have to be considered separately. But aside from the first integral term the remaining parts $(B)\delta y$, $(C)\frac{d\delta y}{dx}$ etc. only depend on the variation of the last value y, which corresponds to $x = a$, and do not involve the nature of the preceding variations; for, to obtain the complete variation of U, one has to put $x = a$ in the found expression, what can actually be done in each part except in the the first, and so in these δy will denote the variation, which is attributed to the last value of y alone and which is completely arbitrary and does not depend on the preceding ones. Hence, even if there would be no integral term, it is perspicuous that from the remaining parts nothing concerning the relation among x and y could be concluded.

§66 But the integral term $\int (A) dx \delta y$ even involves the variations attributed to all the preceding values of y, since it contains the sum of all elements

$(A)dx\delta y$ to result from the variation of each value of y. Therefore, if one value corresponding to x, considered to have a determined value, is varied and is increased by δy, that integral term would only be $= (A)dx\delta y$ and there would not be anything to be summed; but if additionally the following value y' corresponding to $x + dx$ is increased by $\delta y'$ and, having written $x + dx$ instead of x, the function (A) goes over into $(A)'$, the integral term will consist of these two parts

$$(A)dx\delta y + (A)'dx\delta y'.$$

In like manner, if three or more successive values y, y', y'', y''', y'''' etc. are increased by δy, $\delta y'$, $\delta y''$, $\delta y'''$ etc., the integral term will become equal to this expression:

$$(A)dx\delta y + (A)'dx\delta y' + (A)''dx\delta y'' + (A)'''dx\delta y''' + \text{etc.},$$

which series can be imagined continued both backwards to the limit $x = 0$ as forwards to the limit $x = a$.

§67 Therefore, even though the variation δU is restricted to the specific value $x = a$, nevertheless, because of the integral term, it contains all intermediate variations; hence, if for the absolute parts, which are only related to the last term $x = a$, for the sake of brevity we write I, the variation δU will be expressed as follows:

$$\delta U = (A)dx\delta y + (A)'dx\delta y + (A)''dx\delta y'' + (A)'''dx\delta y''' + \text{etc.} + I,$$

which, in order to satisfy the problem, must become equal to zero. But since the variations $\delta y'$, $\delta y''$, $\delta y'''$ etc. do not depend on each other, but each one is arbitrary, that annihilation is only possible if each part vanishes separately, whence is has to be

$$(A) = 0, \quad (A)' = 0, \quad (A)'' = 0, \quad (A)''' = 0\text{etc.},$$

which equations are all contained in the indefinite one $(A) = 0$ or, whatever value is attributed to x, it always has to be $(A) = 0$, and this equation contains the relation among x and y in question.

§68 Lo and behold the simple solution of the propounded problem, in which the relation among x and y is required, from which for the prescribed formula U, after its value had been extend from $x = 0$ to $x = a$, a maximum or minimum value results. Of course, find the variation of the formula U, likewise extended from the limit $x = 0$ to $x = a$, which according to the rules given above must have a form of this kind:

$$\delta U = \int (A) dx \delta y + (B) \delta y + (C) \frac{d\delta y}{dx} + (D) \frac{dd\delta y}{dx^2} + \text{etc.},$$

and hence from the integral term $\int (A) dx \delta y$ alone the relation among x and y will be defined in such a way that $(A) = 0$, but the remaining parts, since they only affect the last value of y, do not contribute anything to the indefinite relation among x and y, which is desired.

§69 Nevertheless, these last parts can serve to obtain a more precise definition of the found relation; for, parts of this kind only enter additionally, if in the integral term $\int (A) dx \delta y$ the function (A) involves the ratio of the differentials $\frac{dy}{dx} = p$ or even the ratios of the higher order differentials, i.e. $q = \frac{dp}{dx}$, $r = \frac{dq}{dx}$ etc. But whenever this happens, the equation $(A) = 0$ will be a differential equation of first or higher order; and hence the relation among x and y in question is just found after one or more integrations. But since each integration introduces an arbitrary constant, this way one will get to a vague finite equation and now there is a new question, how these arbitrary constants must be determined for the value of U to result as a maximum or minimum. For, since each determination of those constants already enjoys the property of the maximum or minimum, here we still have an investigation of the maximum of all maxima or minimum of all minima left.

§70 Therefore, to solve this new problem, one can use those parts not affected by the integral sign. Of course, the constants introduced by integration have to be determined in such a way that for $x = a$ the coefficients of δy, $\frac{d\delta y}{dx}$, $\frac{dd\delta y}{dx^2}$ etc. all vanish, or that in this case these conditions are fulfilled:

$$(B) = 0, \quad (C) = 0, \quad (D) = 0 \quad \text{etc.}$$

Further, since both limits $x = 0$ and $x = a$ can be permuted, also for $x = 0$ it has to be $(B) = 0$, $(C) = 0$, $(D) = 0$ etc. For, even though the parts enforcing this are not contained in our expression, they are nevertheless to be considered to be contained in the integral term.

§71 From the same principles one can even solve problems which I referred to the relative method in my book; indeed these problems can be formulated in general as follows:

Among all relations by which x and y are defined and which enjoy the common property that for the formula $\mathfrak{U}$ for $x = a$ they exhibit the same value, to determine the relation, from which the formula U, if it is extended from $x = 0$ to $x = a$, has the maximum or minimum value.

Therefore, here the variations attributed to each value of y are not arbitrary but are to be constituted in such a way that $\delta\mathfrak{U}$, if its value is extended from $x = 0$ to $x = a$. But then the nature of maxima and minima also demands that for the same extension (from $x = 0$ to $x = a$) as before we have $\delta U = 0$.

§72 Therefore, by the method explained before find the variation to be extended from $x = 0$ to $x = a$ so of the formula $\mathfrak{U}$, which must be in common, as of the formula U, which must become a maximum or a minimum; and the relation among x and y in question is to be investigated from the combination of the two equations $\delta\mathfrak{U} = 0$ and $\delta U = 0$. But these variations will be found expressed this way:

$$\delta\mathfrak{U} = \int (\mathfrak{A}) dx\delta y + (\mathfrak{B})\delta y + (\mathfrak{C})\frac{d\delta y}{dx} + (\mathfrak{D})\frac{dd\delta y}{dx^2} + \text{etc.},$$
$$\delta U = \int (A) dx\delta y + (B)\delta y + (C)\frac{d\delta y}{dx} + (D)\frac{dd\delta y}{dx^2} + \text{etc.},$$

where for the terms without an integral sign the same things are to be remarked as above; and hence one has to derive the relation among x and y in question only from the integral terms.

§73 Hence we will obtain the following two equations:

$$(\mathfrak{A})\delta y + (\mathfrak{A})'\delta y' + (\mathfrak{A})''\delta y'' + (\mathfrak{A})'''\delta y''' + \text{etc.} = 0,$$
$$(A)\delta y + (A)'\delta y' + (A)''\delta y'' + (A)'''\delta y''' + \text{etc.} = 0,$$

in the first of which equations the assumption of the variations δy, $\delta y'$, $\delta y''$ etc. corresponding to the prescribed condition is defined, which is then introduced into the other will manifest the relation in question. Therefore, all variations δy, $\delta y'$, $\delta y''$ etc. except for one can be considered as arbitrary, which one single non-arbitrary variation is to be defined from the first equation. Now it is already evident, after one had already been assumed in such a way that the first equation is satisfied, that then the other is also satisfied at the same time, if one puts $(A) = n(\mathfrak{A})$, taking an arbitrary constant for n.

§74 Therefore, the propounded problem is resolved by this equation:

$$\alpha(A) + \beta(\mathfrak{A}) = 0,$$

having taken arbitrary constants for α and β. But the same solution would have resulted, if among all relations among x and y one had to find that one, whence the formula $\alpha U + \beta\mathfrak{U}$ would have the maximum or minimum value; from this it is at the same time understood that the two propounded formulas $\mathfrak{U}$ and U can be permuted and all the results I mentioned in my book hence become a lot more perspicuous. For, matters will be similar, if not only one formula $\mathfrak{U}$ but several ones must be in common; and hence, having constituted the foundations of the *calculus of variations*, all problems of this kind will be solved most easily and quickly.